Two-page abstract + handout below

The Paraconsistent Numbers and the Set Theory Implied in the Cappadocian Trinitarian Doctrine

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An intuition of pseudo-natural numbers. The Cappadocian doctrine of Trinity is intrinsically paraconsistent. Several Byzantine authors have discussed its paraconsistent features explicitly. For instance, Evagrius Ponticus (345–399) stated: "The numerical triad is followed with the tetrad, but the Holy Trinity is not followed with the tetrad. It is not, therefore, a numerical triad. The numerical triad is preceded with the dyad, but the Holy Trinity is not preceded with the dyad. Thus, it is not a numerical triad" (*Gnostic Chapters* VI, 11-12).

From any modern set-theoretical viewpoint, the main problem of the "set theory" and the very idea of number possibly implied in such statements is the complete absence of the pairs. Indeed, there were several less Orthodox interpretations of Trinity where some kind of pairing was provided, but the Byzantine mainstream throughout the whole *Byzantinische Jahrtausend* was following the path specified by Evagrius (and, before him, his teacher Gregory of Nazianzus). The absolute prohibition of any kind of pairing results into inapplicability of the axiom of Pairing (and, therefore, the axiom of Infinity either) and, what is the most important, any standard set-theoretical definition/interpretation of the notion of number such as that of von Neumann (based on the notion of ordered pair). Of course, this means that, in the framework of von Neumann's approach (and taking the side of Couturat and Zermelo in their discussion with Poincaré), the Trinity is not the number three and even not a number at all. However, this conclusion sounds counterintuitive. And, indeed, in the framework of Poincaré's understanding of number as an "intuition du nombre pur" [1:122], there is no problem in acknowledging that the root "Tri-" in "Trinity" has something to do with the number three… or, at least, *some kind* of number three. Our present task is to define what exactly kind of numbers is meant. In the presently available studies in paraconsistent mathematics there is no description of such mathematical objects [2–3]. I would propose in advance to call the kind of numbers we are looking for "pseudo-natural numbers".

Ternary exclusive OR and the notion of pseudo-ordered pair. The Trinity does not allow pairing because it is not governed with the ordinary exclusive disjunction. The latter regulates the choice between such propositions as "this hypostasis is the Father" or "that hypostasis is the Father". If there are three hypostases, this choice is to be repeated. The table of truth-values of the function corresponding to the ordinary exclusive disjunction shows that this connective, even though preventing any two hypostases from being both the Father, allows being the Father to the three hypostases simultaneously. Somewhere in the Christian Orient such a conclusion was accepted—but not in Byzantine Patristics. To exclude this possibility, we have to use a quite different connective, the so-called ternary exclusive OR ($\veebar^3$), where the choice is presumed to be performed directly from the three and without the choices within the pairs at all. This connective has been at first described by Emil Priest in 1941 but remained almost unstudied until recently [4].

The ternary exclusive OR, however, does not allow constructing the pairs. This, in turn, prevents our numbers in the Trinity from forming any kind of row, that is, an ordered consequence. From the mainstream Byzantine viewpoint (and unlike, among others, the different Western *Filioque* doctrines), there are not, in the Trinity, "the first", "the second", and "the third". An early formulation of this statement is articulated by Severianus of Gabala in the early 5th century, but then, such authors as Nicephorus Blemmydes (1250s) and especially Gregory Palamas (1330s) and Joseph Bryennios (early 15th cent.) have further elaborated on it in the context of the polemics against the *Filioque*.

The famous formula of Gregory of Nazianzus "the monad having from the beginning moved to the dyad stayed at the triad" (*Sermon* 29:2) was interpreted by a part of Byzantine authors (including Maximus the Confessor) epistemologically—as applied to our understanding of the Trinity but not to the Trinity *per se*, whereas another part of them (including, most probably, Gregory himself) understood it ontologically—as pertinent to the Trinity *per se*. However, the monad, the dyad, and the triad of this formula *in its ontological reading* were never identified with specific hypostases. This scheme of the movement of the monad was absolutely symmetrical *vis-à-vis* the three hypostases.

We can ask, however, what means "dyad" in this scheme, if the Trinity does not allow pairing? I would name the implied logical object the pseudo-ordered pair.

The pseudo-ordered pair is to be defined as a paraconsistent generalisation of Kuratowski's standard definition of the ordered pair. In a set of n elements, there is one element chosen to be the first, a; the remaining elements (designed with the letter b with an appropriate index) are in amount of n–1. Thus, the pseudo-ordered pair is

$$\left(a, \bigwedge_{n-1} b_{n-1}\right) =: \bigwedge_{n-1} \{\{a\}, \{a, b_{n-1}\}\}$$

The above formula describes a paraconsistent conjunction: it does not design n–1 pairs, but only a unique pair with n–1 "second" elements.

For the case of the Trinity, $n = 3$. In the contexts irrelevant to the μοναρχία of the Father (the doctrine of the Father as the unique beginning and "the source" of the Trinity), each element (hypostasis) of the Trinity is to be chosen as the first. In the contexts of μοναρχία, where only the Father is the first (e.g., in the *Filioque* polemics), this formula states that the Spirit is not the third after the Son who is the second, but both are equally "the second ones".

The pseudo-natural numbers we were looking for are formed by the pseudo-ordered pairs when all possible choices of the first element are made simultaneously—in an accordance with Severian of Gabala saying that "the divine nature does not have an order—not as disordered but as being beyond any order" (Ps.-John Chrysostom, *Hom. in Gen. 24:2*, ch. 2).

Note: Without being infinite (or "transfinite"), the pseudo-natural numbers imply a difference between their ordinality and cardinality. According to the formal definition above, the ordinality—or rather the "pseudo-ordinality"—is n, whereas the cardinality is 1. To increase the cardinality, we need to allow plurality of the first elements of the pseudo-ordered pair, a, that means allowing plurality of the pairs. This is inapplicable to the Trinity due to the principle of the μοναρχία of the Father. The plurality of a's is not paraconsistent. Thus, the cardinality corresponds to the consistent constituent of the pseudo-natural number (the number of the pseudo-ordered pairs involved), whereas its pseudo-ordinality is paraconsistent.

To encompass the doctrine of uncreated divine energies, the above definition of the pseudo-natural numbers must be generalised over the transfinite objects (that I would call "pseudo-ordinal transfinite numbers").

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References

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Evagrius' summary of the Cappadocian Triadological "mathematics":

VI, 11. La triade numérique est accompagnée d'une tétrade, mais la Trinité sainte n'est pas accompagnée d'une tétrade ; elle n'est donc pas une triade numérique.	ܝܐ — ܠܬܠܝܬܝܘܬܐ ܕܡܢܝܢܐ ܠܘܝܐ ܪܒܝܥܝܘܬܐ. ܠܬܠܝܬܝܘܬܐ ܕܝܢ ܩܕܝܫܬܐ ܠܐ ܠܘܝܐ ܐܪܒܝܥܝܘܬܐ. ܠܝܬܝܗ̇ ܗܟܝܠ ܬܠܝܬܝܘܬܐ ܕܡܢܝܢܐ܀
12. La triade numérique est précédée d'une dyade, mais la Trinité sainte n'est pas précédée d'une dyade ; elle n'est pas, en effet, une triade numérique.	ܝܒ — ܠܬܠܝܬܝܘܬܐ ܕܡܢܝܢܐ܇ ܩܕܝܡܐ ܬܪܝܢܘܬܐ. ܠܬܠܝܬܝܘܬܐ ܕܝܢ ܩܕܝܫܬܐ܇ ܠܐ. ܩܕܝܡܐ ܬܪܝܢܘܬܐ. ܠܝܬܝܗ̇ ܓܝܪ ܬܠܝܬܝܘܬܐ ܕܡܢܝܢܐ܀
13. La triade numérique est constituée par addition d'unités sans substance ; mais la Trinité bienheureuse, ce n'est pas par addition de telles unités qu'elle est constituée ; elle n'est donc pas une triade qui soit avec nombres.	ܝܓ — ܠܬܠܝܬܝܘܬܐ ܕܡܢܝܢܐ܇ ܒܪܘܟܒܐ ܕܚܕ ܚܕ ܕܠܐ ܩܢܘܡ ܡܬܩܝܡܐ. ܬܠܝܬܝܘܬܐ ܕܝܢ ܛܘܒܬܢܝܬܐ܇ ܠܘ ܒܪܘܟܒܐ ܕܚ̈ܕܝܐ ܕܐܝܟ ܗܠܝܢ ܡܬܩܝܡܐ. ܠܝܬܝܗ̇ ܗܟܝܠ ܬܠܝܬܝܘܬܐ ܕܡܢܝܢܐ܀

A. Guillaumont, *Les six centuries des "Kephalaia Gnostica" d'Évagre le Pontique. Édition critique de la version syriaque commune et édition d'une nouvelle version syriaque, intégrale, avec une double traduction française*, Patrologia orientalis, t. 28, f. I, No. 134; Turnhout: Brepols, 1985 [first publ. 1958], 221, 223 (cf., for S_1, pp. 220, 222)

Two kinds of exclusive disjunction: $\oplus$ binary, $\underline{\vee}^3$ ternary ($\underline{\vee}^n$ *n*-ary):

No difference at n=2 (and any other even number):

φ_1	φ_2	$\varphi_1 \oplus \varphi_2$	$\varphi_1 \underline{\vee}^2 \varphi_2$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	F	F

A difference at n=3 (and any other odd number):

φ_1	φ_2	φ_3	$(\varphi_1 \oplus \varphi_2) \oplus \varphi_3$	$\underline{\vee}^3 (\varphi_1, \varphi_2, \varphi_3)$
T	T	T	T	F
T	T	F	F	F
T	F	T	F	F
T	F	F	T	T
F	T	T	F	F
F	T	F	T	T
F	F	T	T	T
F	F	F	F	F

Casimir Kuratowski's definition of the ordered pair (a, b) (where a is the first element and b is the second) is the set $\{\{a\},\{a, b\}\}$, where$\{a, b\}$ is the unordered set (pair) formed with the same elements.

Johann von Neumann's definition of the natural numbers based on the notion of ordered pair:

- The number 0 is defined as the empty set { },
- The successor function is defined as $S(a) = a \cup \{a\}$ for every set a,
- Each natural number is equal to the set of all natural numbers less than it:

  $0 = \{\ \}$,

  $1 = 0 \cup \{0\} = \{0\} = \{\{\ \}\}$,

  $2 = 1 \cup \{1\} = \{0, 1\} = \{\{\ \}, \{\{\ \}\}\}$,

  $3 = 2 \cup \{2\} = \{0, 1, 2\} = \{\{\ \}, \{\{\ \}\}, \{\{\ \}, \{\{\ \}\}\}\}$,

  $n = n{-}1 \cup \{n{-}1\} = \{0, 1, \dots, n{-}1\} = \{\{\ \}, \{\{\ \}\}, \dots, \{\{\ \}, \{\{\ \}\}, \dots\}\}$, etc.

According to **Poincaré**'s approach, the above is an interpretation but not a definition. Anyway, for the row of natural numbers, the existence of ordered pairs is a *conditio sine qua non*.

Definitions of pseudo-ordered pair (the two first generalisations of the notion of ordered pair):

In a set of n elements, there is one element chosen to be the first, a; the remaining elements (designed with the letter b with an appropriate index) are in amount of n–1. Thus, the pseudo-ordered pair is

$$\left(a, \bigwedge_{n-1} b_{n-1}\right) = \bigwedge_{n-1} \{\{a\}, \{a, b_{n-1}\}\} \tag{1}$$

The above formula is paraconsistent: it does not design n–1 pairs, but only a unique pair with n–1 "second" elements.

For the case of the Trinity, $n = 3$.

For the μοναρχία of the Father, the model above is applicable immediately (the Father is always the first, the element a).

For the general case, however, the pseudo-natural numbers we are looking for are formed by the pseudo-ordered pairs when all possible choices of the first element are made simultaneously—in an accordance with Severian of Gabala saying that Οὐ γάρ ἔχει τάξιν ὁ Θεός, οὐχ ὡς ἄτακτος, ἀλλ' ὡς ὑπέρ τάξιν ὤν ("For God does not have an order—not as disordered but as being beyond any order"; Ps.-John Chrysostom, *Hom. in Gen. 24:2*, ch. 2; *PG* 56, 555):

$$\left(\bigwedge_{n} (a_i, a_j)\right) = \bigwedge_{n} \left\{\{a_i\}, \{a_i, a_j\}\right\} \tag{2}$$

where $0 < i \neq j \leq n$.

The pseudo-natural numbers according to definition (1) have a consistent (a) and a paraconsistent (b) parts. The pseudo-natural numbers according to definition (2) are completely paraconsistent.

Further generalisations (necessary for formalisation of the notions of uncreated energies and deification of created beings) would require different kinds of infinite (incl. transfinite) numbers and/or uncountability.